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Numerical Solutions of Fractional Volterra-Fredholm Integro-Differential Equations

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ABSTRACT

Fractional Volterra-Fredholm integro-differential equations (FVFIDEs) have gained increasing attention among researchers over the last decades because of their usefulness in modelling physical phenomena in engineering and technology. Although collocation, Homotopy Perturbation, Adomian Decomposition and related methods have been applied to these problems, many existing approaches involve high computational cost or reduced accuracy for difficult cases. This study therefore applied the Standard Collocation Method (SCM), Least Squares Method (LSM) and Akbari-Ganji Method (A-GM) to obtain numerical solutions of FVFIDEs. The general problem considered is given by:

$$D^{\alpha} y(x) = \int_0^x k(x,t) y(x) + \lambda \int_0^b k(x,t) y(x) + f(x) \quad ;$$

$y^k(0) = \phi_k$; $k = 0,1,2 \ldots n-1$. where D^{α} *denotes the fractional-order derivative of y(x). Orthogonal polynomials were adopted as basic functions for the general class of FVFIDEs. Approximate solutions were assumed in polynomial form and substituted into the governing equations. The resulting equations were collocated at equally spaced interior points, producing systems of linear algebraic equations. These systems were solved for the unknown coefficients, which were then substituted back into the trial solutions to obtain the required approximations. Absolute estimated errors were computed to assess the accuracy and comparative performance of the methods. The findings showed that SCM, LSM and A-GM successfully solved both linear and nonlinear FVFIDEs. The computed absolute errors were small, indicating close agreement between the numerical and exact solutions. LSM and A-GM performed better than SCM, while the results obtained compared favourably with those reported in the literature. The study concluded that SCM, LSM and A-GM are reliable alternative methods for solving FVFIDEs and recommended their adoption for related problems and future studies.*

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INTRODUCTION

The fractional calculus is a name for the study of integrals and derivatives of arbitrary order which unify and generalized the notions of integer order differentiation and integration as discussed by Podlubny (1999). Fractional Calculus is a field of mathematical analysis that studies the several different possibilities of defining real number powers or complex powers of the differentiation operator.

$$D = D^{\alpha} f(x) = \frac{d^{\alpha}}{dx^{\alpha}} f(x) \Rightarrow \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}, \quad (1)$$

and of integration operator

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$$J^{\alpha} f(x) = \frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)} x^{n+\alpha}, \qquad (2)$$

where $f(x) = x^n$.

The term powers used in the above context is referred to iterative application of a linear operator D, to as a function f, that is repeatedly composing D with itself as discussed by Kilbas et al. (2006) and Wazwaz (2011),

$$D^2(f) = (DoD)(f) = D(D(f)) \qquad (3)$$

For example, one may ask for a meaningful interpretation of = as an analogue of the functional square for the differentiation operator, which is, an expression for some linear operator that when applied twice to any function will have the same effect as differentiation (Kilbas et al., 2006).

More generally, as of fractional differential equation is given as:

$$D^{\alpha} f(x) = \frac{d^{\alpha}}{dx^{\alpha}}(x, y(x)) \qquad (4)$$

It can be said that a Fractional Differential Equation is a generalization of differential equation through the application of fractional calculus. A fractional order system means a system described by a fractional differential equation or by a system of such equation. Fractional calculus which might be considered as an extension of classical calculus is as old as calculus itself (Podlubny, 1999). Fractional differential equation have been used to describe many scientific phenomena in engineering, physics, chemistry, image processing, visco elasticity, electromagnetic, biomedical engineering, statistic and control theory (Kilbas et al., 2006).

Finding the exact or analytic solution of fractional order differential equation is not an easy task. Hence several different numerical solution techniques have been developed for the approximate solution of these types of equations. Some of the well known numerical techniques might be listed as, Finite Difference Method(FDM), Iterative Decomposition Method(IDM), Homotopy perturbation and Homotopy analysis method, Collocation method, Adomian Decomposition Method(ADM) and Differential Transform Method(DTM).

Derivatives of non-integer order are defined in different ways, e.g., Caputo and Riemann-Liouville (Kilbas et al., 2006). In this work, we are concerned with the numerical solution of Fractional Integro Differential Equations by Galerkin Method Using Chebyshev Polynomial as a basic function (Adebisi et al., 2021a). The general form of fractional integro diiferential equation is given as:

$$D^{\alpha} y(x) = f(x) + \int_0^1 k(x,t) y(t) dt \qquad 0 \le x, t \le 1 \qquad (5)$$

with initial condition $y(0) = y_0$

Where indicates the Caputo- Fractional derivative of y(x) and k(x,t), f(x) are smooth functions, x and t are real variables defined on the interval and y(x) is the unknown function to be determined. It has been observed that most linear and non-linear, differential and integro-differential equations which are difficult to solve analytically can be handled numerically with a good and appropriate numerical method, when deployed appropriately produces a good approximate solution (Falade & Taiwo, 2023; Adebisi et al., 2021a; Wazwaz, 2011). The common numerical methods used to solve FDEs includes Collocation Method, Variation Iteration Method, Adomian Decomposition Method, Differential Transform Method, Homotopy Analysis Method, Least Squares Method (LSM), Laplace Decomposition Method, Iterative Decomposition Method, Galerkin Method, and Successive Substitution Method to mention but a few.

Collocation and Least Squares Methods are used to solve Volterra-Fredholm fractional integro-differential equations. As discussed by Podlubny (1999), the field of fractional calculus is as old as calculus itself, but over the last decades the usefulness of this mathematical theory in applications as well as its merits in pure

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mathematics has become more and more evident. Most formulations of mathematical models of physical problems lead to linear or nonlinear differential equations. Many of these differential equations do not have analytical solution. To solve these types of equations, we have to employ numerical collocation methods.

Fractional Calculus, which is the branch of mathematical analysis that studies the possibility of taking fractional powers of differential operators is deduced from the fundamental theorem of classical calculus.

$$D \equiv \frac{d}{dt} f(t) = f'(t) \tag{6}$$

where D represents fractional operator of $f(t)$.

If $f(t) = t^r$, then

$$\frac{d^n}{dt^n} t^r = \frac{r!}{(r-n)!} t^{r-n} \tag{7}$$

Rewriting equation (7) using gamma function, gives

$$\frac{d^n}{dt^n} t^r = \frac{\Gamma(r+1)}{\Gamma(r-n+1)} t^{r-n}, \qquad r \geq 0 \tag{8}$$

Hence, if $r = 1$ and $n = \frac{1}{2}$, then the half derivative of the function t is obtained as

$$\frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}} t^1 = \frac{\Gamma(1+1)}{\Gamma(1-\frac{1}{2}+1)} t^{1-\frac{1}{2}} = \frac{1!}{\Gamma(\frac{3}{2})} t^{\frac{1}{2}} = \frac{2t^{\frac{1}{2}}}{\sqrt{\pi}} = 2\sqrt{\frac{t}{\pi}} \tag{9}$$

The role of differential equations cannot be over emphasized in the representation and interpretation of real life problems as most of them are formulated using differential equations. They play relevant roles in the field of science and engineering technology. Fractional differential equations (differential equations where the derivatives are of fractional order) also play vital role in every field of science. Kilbas et al. (2006) revealed that fractional differential equations have a lot of applications in many problems in science and engineering technology. Much earlier, Podlubny (1999) stated that the Fluid Dynamic Traffic Model (FDTM) was developed based on fractional differential calculus. Fractional calculus is applicable in the modeling of physical occurrences such as polymers and the description of deformation and flow properties of rock materials (Kilbas et al., 2006; Wazwaz, 2011).

Also, denoting J_a as an integration operator that maps a function f onto its primitive and centered at 'a' then:

$$J_a f(t) \equiv \int_a^x f(t)dt \qquad a < t < b \tag{10}$$

If $n \in N$, the symbols D^n and J^n are used to denote the n-fold iterates of D and J_a respectively.

Fractional differential equation, also known as extraordinary differential equation is a generalization of differential equations with the application of fractional calculus. According to Podlubny (1999), fractional calculus does not mean the calculus of fractions or fractions of any calculus (differential or integral) but the theory of integrals and derivatives of arbitrary order, which unify and generalize the notions of integer-order differentiation and -fold integration. Generally, it is denoted by where and are the terminal limits relating to the operation and the order of the derivative. A fractional differential equation of order , is an equation of the form:

$$D^\alpha y(x) = f(x, y(x)) \qquad a < \alpha < b \tag{11}$$

The concept of fractional differential equations has been defined by many researchers, but two commonly used are that of Riemann liouville and Caputo definitions. The Riemann-Liouville differential operator is denoted as D^α while the Caputo differential operator is given as D_*^α.

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Fractional Calculus is a generalization of ordinary differentiation and integration to an arbitrary order. This class of calculus has gained tremendous attention in recent years as researchers utilize it in solving problems in the field of science and engineering Technology. It is worth noting that most problems of fractional integro differential equation cannot be solved analytically. Hence good approximate solutions using numerical methods are usually employed to solve such problems. To this end, many numerical techniques have been advanced to include Homotopy perturbation method, Adomian decomposition method, Perturbation Collocation Method , Variational Iterative Method , Laplace Decomposition Method , Iterative Decomposition Method, Galerkin Method , Collocation Method , Successive Substitution Method, to mention but a few. According to Adebisi et al. (2021a), orthogonal polynomial functions are very useful in many problems of approximation theory, mathematical and numerical analysis and their applications. Orthogonal polynomials are special functions of mathematical physics which are important research subject for pure and applied mathematics, and their applications.

The numerical treatment of fractional differential and integro-differential equations has continued to attract attention because such models describe memory-dependent processes that occur in science, engineering and technology. Podlubny (1999), Kilbas et al. (2006) and Wazwaz (2011) provided important theoretical and computational foundations for fractional differential, integral and integro-differential equations. In the same direction, Alkhalissi et al. (2021) used generalized Gegenbauer-Humbert polynomials to develop an operational matrix approach for fractional differential equations, while Sadabad et al. (2020) applied the Laplace transform to fractional Sturm-Liouville problems. These studies show that polynomial and transform-based procedures remain useful tools for reducing fractional models to solvable algebraic forms.

Several recent studies have focused on collocation-based approaches for Volterra, Fredholm and Volterra-Fredholm integro-differential equations. Falade (2019) used an exponentially fitted collocation approximation technique for integro-differential equations, and Falade and Taiwo (2023) further applied an exponentially fitted collocation algorithm to nth-order Fredholm integro-differential equations. Related collocation schemes were reported by Aduroja et al. (2023), Ajileye and Amoo (2023), Etuk et al. (2021), Shior et al. (2022), and Oyedepo et al. (2023), where polynomial bases such as Chebyshev, Legendre, Laguerre and monic Chebyshev polynomials were used to transform the governing equations into systems of algebraic equations. These works confirm that collocation techniques are simple, reliable and accurate for the numerical solution of integro-differential equations.

Least squares, Galerkin and related polynomial approximation methods have also been applied successfully to fractional and higher-order integro-differential problems. Oyedepo et al. (2022) solved fractional integro-differential equations using the least square Bernstein method, whereas Bello et al. (2024a) applied the least squares method with shifted Chebyshev polynomials to fourth-order Volterra integro-differential equations. Bello et al. (2024b) also used the Galerkin method with Chebyshev polynomial basis for multi-order fractional differential equations. In addition, Nour and Muna (2020), Leyla et al. (2023), Toma and Postavaru (2023), Otaide and Oluwayemi (2024), Ogunwobi et al. (2025), and Falade et al. (2025) demonstrated the effectiveness of Lagrange polynomials, block pulse functions, Fibonacci polynomials, variational iteration-collocation procedures and constructed orthogonal polynomials for fractional Volterra-Fredholm and related integro-differential equations.

More directly, the body of work associated with F. A. Adebisi has strengthened the use of orthogonal-polynomial, collocation, Galerkin and least-squares procedures for integro-differential and fractional models. Adebisi et al. (2019) employed a multiple perturbed collocation tau method for nonlinear integro-differential equations, while Adebisi et al. (2021a) applied Chebyshev polynomial basis functions

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within a Galerkin framework for Volterra integro-differential equations. Adebisi et al. (2021b) further combined Galerkin and perturbed collocation methods for a class of linear fractional integro-differential equations. In recent extensions, Adebisi and Okunola (2025) used the Chebyshev least squares method for Volterra-Fredholm integro-differential equations, and Adebisi et al. (2025) applied Chebyshev least squares approximation to fractional-order differential equations. These studies are closely related to the present application of Standard Collocation, Least Squares and Akbari-Ganji Methods to FVFIDEs.

The Akbari-Ganji Method has also been employed for nonlinear and fractional differential models because of its ability to obtain approximate solutions with fewer computational steps. Akbari et al. (2020) applied the method to non-isothermal and non-adiabatic mixed-flow reactors, while Akbari et al. (2023) developed a related approach for complicated nonlinear differential equations in engineering and basic sciences. Attar et al. (2022) presented an analytical application of the Akbari-Ganji Method to fractional differential equations, and Uwaheren et al. (2022) applied the method to Volterra integro-differential equations. The present study therefore builds on these contributions by applying Standard Collocation, Least Squares and Akbari-Ganji Methods to fractional Volterra-Fredholm integro-differential equations and by comparing their absolute estimated errors.

PRELIMINARY

Integro-Differential Equations:

Integro-differential equations arise when initial value problems are transformed into integral equations. They involve the unknown function and its derivative both inside and outside an integral. At least one limit of integration is usually variable, similar to Volterra integral equations. These equations combine differential and integral operators and require initial conditions to determine particular solutions as discussed by Wazwaz (2011).

Fredholm Integro-differential Equation:

If the limit of integration are fixed then the Integro-differential equation is called Fredholm Integro-differential equation. Which is of the form

$$u^{(k)}(x) = f(x) + \lambda\int_a^b K(x;t)u(t)dt, \quad u^{(k)} = \frac{d^k u}{dx^k} \tag{12}$$

where and are lower and upper limits of integration which are constants as seen in Wazwaz (2011).

Volterra Integro-differential Equations:

If at least one of the limits is a variable then Integro-differential equation is called Volterra integral-differential equation. Which is of the form

$$u^{(k)}(x) = f(x) + \lambda\int_a^x K(x;t)u(t)dt, \quad u^{(k)} = \frac{d^k u}{dx^k} \tag{13}$$

Where one of the limits is a variable.

Volterra-Fredholm Integro-differential Equation:

The Volterra-Fredholm integro-differential equations occur in the same manner as Volterra-Fredholm integral equations with one or more of ordinary derivatives in addition to the integral operators (Wazwaz, 2011). They occur in two forms namely

$$u^{(k)}(x) = f(x) + \lambda\int_a^x K_1(x,t)u(t)dt + \lambda\int_a^x K_2(x,t)u(t)dt \tag{14}$$

and

$$u^{(k)}(x) = f(x) + \lambda\int_a^x\int_a^x K(x,t)u(t)dt \tag{15}$$

where $f(x,t)$ are analytic functions in the interval (a, b). $K_i(x,t)$ are the kernel of the function, and λ_i are arbitrary parameters

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METHODOLOGY

Description of standard collocation method:

Here, the step-by-step algorithms of the methods used in this work are presented

Description of Standard Collocated Method

Consider the general class of Volterra-Fredholm Integro-Differential Equation of the form:

$$D^{\alpha} y(x) = \lambda_1 \int_a^x k_1(x,t)u(t)dt + \lambda_2 \int_a^b k_2(x,t)u(t)dt + f(x) \qquad x \in [a,b] \tag{16}$$

subject to conditions:

$$y^k(0) = \phi_k ; k = 1; 2....n \tag{17}$$

where, λ_1 and λ_2 are constants, $k_1(x,t)$ and $k_2(x,t)$ are the kernels of the problem

A single polynomial approximation $y_N(x)$ of the unique solution y(x) of problem (16) and (17), is determined by assuming an approximate solution of the form:

$$y(x) \equiv y_N(x) = \sum_{j=0}^{N} c_j T_j(x) \tag{18}$$

where c_j are unknown constants to be determined and $T_j(x)$ are the Chebyshev polynomials functions derived in the previous section.

Equation (16) is re-written as

$$D^{\alpha} y(x) - \lambda_1 \int_a^x k_1(x,t)u(t)dt - \lambda_2 \int_a^b k_2(x,t)u(t)dt = f(x) \tag{19}$$

On substituting (18) into (19), gives

$$D^{\alpha} \sum_{j=0}^{N} c_j T_j(x) - \lambda_1 \int_0^x (x,t) \sum_{j=0}^{N} c_j T_j(t)dt - \lambda_2 \int_0^b (x,t) \sum_{j=0}^{N} c_j T_j(t)dt = f(x) \tag{20}$$

Equation (20) is collocated at equally spaced interior points $x = x_j$, where

$x_j = a + \dfrac{(b-a)j}{N}, (j = 0,1,2,...,(N+1))$ to gives.

$$D^{\alpha} \sum_{j=0}^{N} c_j T_j(x_0) - \lambda_1 \int_0^x (x,t) \sum_{j=0}^{N} c_j T_j(t_0)dt - \lambda_2 \int_0^b (x,t) \sum_{j=0}^{N} c_j T_j(t_0)dt = f(x_0) \tag{21}$$

$$D^{\alpha} \sum_{j=0}^{N} c_j T_j(x_1) - \lambda_1 \int_0^x (x,t) \sum_{j=0}^{N} c_j T_j(t_1)dt - \lambda_2 \int_0^b (x,t) \sum_{j=0}^{N} c_j T_j(t_1)dt = f(x_1) \tag{22}$$

$$D^{\alpha} \sum_{j=0}^{N} c_j T_j(x_2) - \lambda_1 \int_0^x (x,t) \sum_{j=0}^{N} c_j T_j(t_2)dt - \lambda_2 \int_0^b (x,t) \sum_{j=0}^{N} c_j T_j(t_2)dt = f(x_2) \tag{23}$$

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$$D^{\alpha}\sum_{j=0}^{N}c_jT_j(x_N)-\lambda_1\int_0^x(x,t)\sum_{j=0}^{N}c_jT_j(t_N)dt-\lambda_2\int_0^b(x,t)\sum_{j=0}^{N}c_jT_j(t_N)dt = f(x_N) \quad (24)$$

thus, resulting to $(N+1)$ linear system of algebraic equations which can be expressed in matrix form:

$$Ax=b\Rightarrow\begin{bmatrix} A_{11} & A_{12} & L & A_{1m} \\ A_{21} & A_{22} & L & A_{2m} \\ M & M & M & M \\ A_{n1} & A_{n2} & L & A_{mn} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ M \\ c_4 \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \\ M \\ bn \end{bmatrix} \quad (25)$$

The system of linear $(N+1)$ equations are then solved using Gaussian elimination method or any suitable computer package like Maple 18 to obtain the unknown constants, which are then substituted back into the assumed approximate solution to give the required approximate solution.

Example 1: Standard Collocated Method

Consider the fractional Volterra-Fredholm integro-differential equations.

$$D^{\alpha}u(x)=-6-2x-3x^2-4x^3+\int_0^x(1-xt)y(t)dt+\int_0^1(x-t)y(t)dt \quad Wazwaz(2011) \quad (26)$$

subject to initial conditions (ic)

$$y(0)=2,\, y'(0)=6y''(0)=24 \quad (27)$$

where $\alpha=\dfrac{7}{2}$ and exact solution is $y(x)=2+6x+12x^2$

We assume

$$y_5(x)=\sum_{j=0}^{5}c_jy(x)=c_0+c_1(2x-1)+c_2(8x^2-8x+1)$$

$$+c_3(32x^3-48x^2+18x-1)+c_4(128x^4-256x^3+160x^2-32x+1) \quad a\le x\le b \quad (28)$$

and substituting (28) into (26), we have

$$D^{\alpha}\sum_{j=0}^{5}c_jy(x)=-6-2x-3x^2-4x^3+\int_0^x(1-2xt)\sum_{j=0}^{5}c_jy(t)dt+\int_0^1(x-t)\sum_{j=0}^{5}c_jy(t)dt, \quad (29)$$

$\forall x\in[0,1]$. Applying $\alpha=\dfrac{7}{2}$-derivative on the left-hand side of (27), gives

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$$81920x^5c_5+6144x^4c_4-6144x^4c_5+192x^3c_3-1536x^3c_4+6720x^3c_5$$
$$-8x^2c_2+48x^2c_3-160x^2c_4+400x^2c_5+\frac{3}{2}c_1x-6xc_2+\frac{27xc_3}{2}-24xc_4$$
$$+\frac{75xc_5}{2}-\frac{15c_0}{8}+\frac{15c_1}{8}-\frac{15c_2}{8}+\frac{15c_3}{8}-\frac{15c_4}{8}+\frac{15c_5}{8} \quad (30)$$

Integrating the right-hand side of (28) accordingly, gives

$$-\frac{64x^7a_4}{3}+\frac{640x^7a_5}{3}-\frac{416x^6a_5}{3}-4x^6a_3+\frac{256x^6a_4}{5}-\frac{72x^5a_4}{5}\frac{3}{2}x^5a_2$$
$$-\frac{2}{3}x^4a_1-\frac{512x^8a_5}{7}+a_0x+\frac{790x^4a_5}{3}-\frac{160x^4a_4}{3}+\frac{317x^3a_4}{6}-16x^2a_4+xa_4$$
$$-156x^5a_5-\frac{1}{2}a_0-\frac{1}{6}a_1+\frac{1}{42}a_5 \quad (31)$$

Equation (28) is put into the problem (26) to give

$$-16x^8c_3+128x^8c_4-\frac{68x^8c_5}{7}-\frac{24x^7c_2}{5}+49x^5c_3-161x^5c_4-\frac{384x^9c_4}{7}$$
$$+\frac{2}{3}x^2c_1+\frac{64x^7a_4}{3}-\frac{640x^7a_5}{3}+\frac{416x^6a_5}{3}+4x^6a_3-\frac{256x^6a_4}{5}+\frac{72x^5a_4}{5}$$
$$-\frac{15}{2}x^5a_3+\frac{2}{3}x^4a_1+\frac{512x^8a_5}{7}-2a_0x-\frac{790x^4a_5}{3}+\frac{797x^3a_5}{6}-25x^2a_5$$
$$+156x^5a_5-36.14146667a_4+13.54055000a_3-6.770275002a_2+\frac{1}{2}a_0$$
$$+2.423425001a_1+56.39514883a_5+3x^2+4x^3+2x=0 \quad (32)$$

The source function, $f(x)=-6-2x-3x^2-4x^3$ is included and after some simplifications accordingly gives

$$-\frac{8}{3}x^4c_1+\frac{32x^4c_2}{3}-24x^4c_3-2x^3c_0+2x^3c_1-2x^3c_2+1144x^6c_4$$
$$-1867x^6c_5-x^5c_0+x^5c_1-9x^5c_2+49x^5c_3-161x^5c_4-384x^9c_4+1384x^9c_5+$$
$$+128x^8c_4-\frac{24x^7c_2}{5}+\frac{144x^7c_3}{5}-\frac{544x^7c_4}{3}+\frac{976x^7c_5}{3}-\frac{3}{2}x^6c_1+6x^6c_2$$
$$-192x^{10}c_5+\frac{116x^2c_2}{15}+34x^3c_3-\frac{242x^2c_3}{5}+\frac{512x^4c_4}{3}-258x^3c_4+\frac{3356x^2c_4}{21}-$$
$$+1122x^3c_5+\frac{5422x^2c_5}{21}+\frac{512x^8a_5}{7}-\frac{7363xc_5}{14}-\frac{319xc_4}{10}+\frac{183xc_3}{10}-=0 \quad (33)$$

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Equation (31) is collocated points at $x = 0, 0.2, 0.4, 0.6, 0.8, 1.0$ to get

$$c_0 - c_1 + c_2 - c_3 + c_4 - c_5 = 0 \quad (34)$$

$$-6.95915581c_5 - 0.817450655c_4 + 0.971105280c_3 - 0.192157440c_2 +0.7370133333c_0 - 0.66137600c_1 + 0.041664 = 0 \quad (35)$$

$$-7.7423823c_5 + 0.72752435c_4 + 0.515906560c_3 - 0.693048320c_2 +0.4750933333c_0 - 0.2295040000c_1 + 0.189696 = 0 \quad (36)$$

$$1.1353406c_5 + 0.45826344c_4 - 0.68693248c_3 - 0.319873280c_2 +0.0702400000c_0 + 0.2341760000c_1 + 0.536256 = 0 \quad (37)$$

$$4.6245501c_5 - 1.12709180c_4 - 0.59646720c_3 + 0.911511040c_2 -0.698346667c_0 0.4928639997c_1 + 1.311744 = 0 \quad (38)$$

$$1.51375302c_5 + 0.57752643c_4 + 0.45588246c_3 - 0.16515040c_2 -0.866834673c_0 + 0.1409284397c_1 + 3.31917424 = 0 \quad (39)$$

The equations (33) to (37) are solved simultaneously to obtain the constants values

$$c_0 = 382.6334712\ c_1 = 7.0507242 \quad c_2 = 8.8191650 \quad c_3 = 3.92923565$$
$$c_4 = 8.254993814 c_5 = -1.272329878$$

Substituting the values into the trial solution, gives the required approximate solution

$$y_5(x) = -0.916845e^{-6}x^4 - 0.9995e^{-4}x^4 + 0.9890596x^3 + 1.0048500x^2 +0.6850x + 0.12100e^{-3} \quad (40)$$

Example 2: Standard Collocated Method

Consider the fractional Volterra-Fredholm integro-differential equations.

$$D^{\alpha} y(x) = -x^2 - x^4 - x^6 + \int_0^x (3xt + 4x^2t^2) y(t)dt + \int_0^1 3xt^2 y(t)dt, \quad x \in [0,1] \quad (41)$$

subject to initial conditions (ic)

$$y(0) = y'(0) = 0 \quad (42)$$

where $\alpha = 2.5$ and exact solution is Assuming $y(x) = x^2 + x^3$

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$$y_5(x)=\sum_{j=0}^{5}c_j y(x)=c_0+c_1(2x-1)+c_2(8x^2-8x+1)$$
$$+c_3(32x^3-48x^2+18x-1)+c_4(128x^4-256x^3+160x^2-32x+1)$$
$$a\le x\le b \tag{43}$$

substituted (43) into (41), we have

$$D^{\alpha}\sum_{j=0}^{5}c_j y(x)=-x^2-x^4-x^6+\int_0^x(3xt+4x^2t^2)\sum_{j=0}^{5}c_j y(t)dt+\int_0^1 3xt^2\sum_{j=0}^{5}c_j y(t)dt \tag{44}$$

$\forall x\in[0,1]$ Applying $\alpha=\frac{5}{2}$ -derivative on the left hand side of (41), gives

$$32768x^5c_5+4096x^4c_4-4096x^4c_5+384x^3c_3-3072x^3c_4+13440x^3c_5$$
$$+16x^2c_2-96x^2c_3+320x^2c_4-800x^2c_5-xc_1+4xc_2-9xc_3+16xc_4-25xc_5$$
$$+\frac{3}{4}c_0-\frac{3}{4}c_1+\frac{3}{4}c_2-\frac{3}{4}c_3+\frac{3}{4}c_4-\frac{3}{4}c_5 \tag{45}$$

Integrating the right hand side of (41) accordingly, gives

$$-xa_5-a_1x-3x^2a_2+xa_2-\frac{19}{2}x^3a_3+6x^2a_3-xa_3-\frac{160x^4a_4}{3}+\frac{317x^3a_4}{6}$$
$$-16x^2a_4+xa_4-156x^5a_5-\frac{64x^7a_4}{3}+\frac{640x^7a_5}{3}-\frac{416x^6a_5}{3}-4x^6a_3+\frac{256x^6a_4}{5}$$
$$-\frac{72x^5a_4}{5}-\frac{3}{2}x^5a_2+\frac{15}{2}x^5a_3+x^4a_3-\frac{2}{3}x^4a_1+2x^4a_2+\frac{3}{2}x^3a_2-\frac{1}{2}x^3a_0$$
$$+\frac{1}{2}x^3a_1+x^2a_1-\frac{512x^8a_5}{7}+a_0x+\frac{790x^4a_5}{3}-\frac{797x^3a_5}{6}+25x^2a_5 \tag{46}$$

and these further results to,

$$35a_0x-\frac{1}{15}xa_4+\frac{11}{30}a_4-\frac{1}{2}a_0-\frac{25}{6}a_1+\frac{3}{42}a_5 \tag{47}$$

Equation (45) and (46) are put into the problem (43) to give

$$6x^8c_3+128x^8c_4-\frac{68x^8c_5}{7}-\frac{24x^7c_2}{5}+49x^5c_3-161x^5c_4-\frac{84x^9c_4}{2}+\frac{384x^9c_5}{3}$$
$$+\frac{2}{3}x^2c_1+4/3x^2c_06+49x^5c_3-161x^5c_4-\frac{13x^9c_4}{7}+\frac{384x^5c_5}{3}+\frac{2}{3}x^2c_1+\frac{4}{3}x^2c_0$$
$$+128x^8c_4-\frac{5968x^8c_5}{7}-\frac{24x^7c_2}{5}+\frac{64x^7a_4}{3}-\frac{640x^7a_5}{3}+\frac{416x^6a_5}{3}+4x^6a_3$$

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$$+\frac{72x^5a_4}{5}+\frac{3}{2}x^5a_2-\frac{15}{2}x^5a_3-x^4a_3+\frac{2}{3}x^4a_1-2x^4a_2-\frac{3}{2}x^3a_2+\frac{1}{2}x^3a_0$$
$$+\frac{512x^8a_5}{7}-\frac{14xa_4}{15}+156x^5a_5-36.14146667a_4+13.54055000a_3$$
$$+\frac{1}{2}a_0+2.423425001a_1+56.395\frac{790x^4a_5}{3}+\frac{797x^3a_5}{6}+\frac{19}{2}x^3a_3+\frac{160x^4a_4}{3}=0 \quad (48)$$

The source function, $f(x)=-x^2-x^4-x^6$ is included and after some simplifications accordingly gives

$$\frac{232x^2c_2}{15}-\frac{1192x^2c_3}{15}-c_5+\frac{8504x^2c_4}{21}+x^5c_5-\frac{16568x^2c_5}{21}-96x^7c_4$$
$$-\frac{27x^6c_3}{2}+24x^6c_4-\frac{32x^3x^2c_2}{3}+4x^2xc_5-1120x^4x^2c_5-\frac{1024x^6x^2c_5}{3}$$
$$-400x^2c_5+2xc_1-8xc_2+18xc_3-32xc_4+50xc_5+x^6+x^4+x^2=0 \quad (49)$$

Equation (48) is collocated at points $x=0,0.2,0.4,0.6,0.8,1.0$ to get

$$1.87532c_5-1.5750032c_4+1.875052c_3-1.77532c_2-1.671035c_1$$
$$+1.875632c_1+1.87500c_0=0 \quad (50)$$

$$0.041664+6.64080418c_5-15.50685066c_4+8.066105280c_3$$
$$-3.307157440c_2-2.137986667c_0+2.113624000c_1=0 \quad (51)$$

$$0.189696+1066.140938c_5+21.94212441c_4+27.19090656c_3$$
$$-5.328048320c_2-2.399906667c_0+2.445496000c_1=0 \quad (52)$$

$$0.536256+7058.722820c_5+390.3768635c_4+68.60806752c_3$$
$$-7.754873280c_2-2.804760000c_0+2.809176000c_1=0 \quad (53)$$

$$1.311744+27788.31938c_5+1606.391508c_4+142.0385328c_3$$
$$-10.60348896c_2-3.573346667c_0+2.967864000c_1=0 \quad (54)$$

$$81756.08929c_{5[]}+4422.310715c_4+255.9750000c_3-14.77500000c_2$$
$$-5.041666667c_0+2.375000000c_1=0 \quad (55)$$

The equations (49) to (54) are solved simultaneously to obtain the constants values

$$c_0=0.5937222422e^{-1}c_1=0.5492088673e^{-1},c_2=-0.1306265403e^{-1},$$

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$$c_3 = -0.7945474225e^{-2}, c_4 = 0.6201832962e^{-3}, c_5 = -0.4565902503e^{-4}$$

Substituting the values into the trial solution, gives the required approximate solution

$$y_5(x) = 0.00000087650x^5 - 0.64300001e - 4x^4 + 0.501435e - 2x^3$$
$$+1.0000x^2 + 6.045x + 2.598657 = 0 \quad (56)$$

2 Description of Least Squares Method

Consider the Volterra-Fredholm integro-differential equation

$$D^{\sigma} y(x) + \lambda_1 \int_a^b k(x,t) y(t) dt + \lambda_2 \int_a^b k(u,t) y(t) dt = f(x) \qquad x \in [a,b] \quad (57)$$

and an assume an approximate solution

$$y_N(x) = \sum_{j=0}^{N} c_j y_j(x) = c_0 + c_1(2x-1) + c_2(8x^2 - 8x + 1)$$
$$+c_3(32x^3 - 48x^2 + 18x - 1) + c_4(128x^4 - 256x^3 + 160x^2 - 32x + 1) \quad (58)$$

Substituting equation (58) into (57), gives:

$$D^{\sigma}\left(\sum_{i=0}^{N} a_i T_i^*(x)\right) + \lambda_1 \int_a^b k(x,t)\left(\sum_{i=0}^{N} a_i T_i^*(t)\right) dt$$
$$+\lambda_2 \int_a^b k(x,t)\left(\sum_{i=0}^{N} a_i T_i^*(t)\right) dt = f(x) \qquad x \in [a,b] \quad (59)$$

The Residual form of equation (59) is written as :

$$D^{\sigma}\left(\sum_{i=0}^{N} a_i T_i^*(x)\right) + \int_a^b k(x,t)\left(\sum_{i=0}^{N} a_i T_i^*(t)\right) dt + \int_a^b k(x,t)\left(\sum_{i=0}^{N} a_i T_i^*(t)\right) dt - f(x) = 0 \qquad x \in [a,b] \quad (60)$$

To minimize Residual equation (60) we compute

$$S(c_0, c_1, c_2, \dots c_N) =$$
$$\int_0^1 [D^{\sigma}\left(\sum_{i=0}^{N} a_i T_i^*(x)\right) + \int_a^b k(x,t)\left(\sum_{i=0}^{N} a_i T_i^*(t)\right) dt + \int_a^b k(x,t)\left(\sum_{i=0}^{N} a_i T_i^*(t)\right) dt - f(x) = 0]^2 dt \quad (61)$$

Applying the D^{α} to equation (61), we have

$$S(c_0, c_1, c_2, \dots c_N) =$$
$$\int_0^1 [\sum_{j=0}^{N} c_j \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} t^{j-\alpha} + \int_a^b \sum_{j=0}^{N} c_j \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} t^{j-\alpha} dt + \int_a^b \sum_{j=0}^{N} c_j \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} t^{j-\alpha} dt - f(x)]^2 dt = 0 \quad (62)$$

Equation (62) is minimized by differentiating it partially with respect to $c_0, c_1, c_2 \dots c_N$ and equating to zero gives the $(n+1)$ system of equations:

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$$\frac{\partial S}{\partial c_0} = -2\int_0^1 [c_0 \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} t^{j-\alpha} + \int_a^b \sum_{j=0}^{N} c_j \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} t^{j-\alpha} dt + \int_a^b \sum_{j=0}^{N} c_j \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} t^{j-\alpha} dt - f(x)] = 0 \quad (63)$$

$$\frac{\partial S}{\partial c_1} = -2\int_0^1 [c_0 \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} t^{j-\alpha} + \int_a^b \sum_{j=0}^{N} c_j \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} t^{j-\alpha} dt + \int_a^b \sum_{j=0}^{N} c_j \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} t^{j-\alpha} dt - f(x)]t = 0 \quad (64)$$

$$\frac{\partial S}{\partial c_2} = -2\int_0^1 [c_0 \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} t^{j-\alpha} + \int_a^b \sum_{j=0}^{N} c_j \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} t^{j-\alpha} dt + \int_a^b \sum_{j=0}^{N} c_j \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} t^{j-\alpha} dt - f(x)]t^2 = 0 \quad (65)$$

$$\frac{\partial S}{\partial c_N} = -2\int_0^1 [c_0 \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} t^{j-\alpha} + \int_a^b \sum_{j=0}^{N} c_j \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} t^{j-\alpha} dt + \int_a^b \sum_{j=0}^{N} c_j \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} t^{j-\alpha} dt - f(x)]t^n = 0 \quad (66)$$

The system of linear equations (63) to (65) obtained are solved using a computer software, maple 18 to obtain the $(n+1)$ constant coefficients. The values of the constants are substituted into the trial solution to obtain the required approximate solution.

Example 1: Least Square Method

Consider the fractional Volterra-Fredholm integro-differential equations.

$$D^{\alpha}u(x) = -6 - 2x - 3x^2 - 4x^3 + \int_0^x (1-xt)y(t)dt + \int_0^1 (x-t)y(t)dt \quad (67)$$

subject to initial conditions (ic)

$y(0) = 2, y'(0) = 6y''(0) = 24$ (68)

where $\alpha = \frac{7}{2}$ and exact solution is $y(x) = 2 + 6x + 12x^2$ We assume

$$y_5(x) = \sum_{j=0}^{5} c_j y(x) = c_0 + c_1(2x-1) + c_2(8x^2 - 8x + 1)$$

$$+c_3(32x^3 - 48x^2 + 18x - 1) + c_4(128x^4 - 256x^3 + 160x^2 - 32x + 1) + a \le x \le b \quad (69)$$

and substituting (69) into (67), we have

$$D^{\alpha}\sum_{j=0}^{5} c_j y(x) = -6 - 2x - 3x^2 - 4x^3 + \int_0^x (1-2xt)\sum_{j=0}^{5} c_j y(t)dt + \int_0^1 (x-t)\sum_{j=0}^{5} c_j y(t)dt, x \in [0,1] \quad (70)$$

Applying $\alpha = \frac{7}{2}$ -derivative on the LHS of (70), gives

$$81920x^5c_5 + 6144x^4c_4 - 6144x^4c_5 + 192x^3c_3 - 1536x^3c_4 + 6720x^3c_5 - 8x^2c_2$$

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$$+48x^2c_3 - 160x^2c_4 + 400x^2c_5 + \frac{3}{2}c_1x - 6xc_2 + \frac{27xc_3}{2} - 24xc_4 + \frac{75xc_5}{2}$$
$$-\frac{15c_0}{8} + \frac{15c_1}{8} - \frac{15c_2}{8} + \frac{15c_3}{8} - \frac{15c_4}{8} + \frac{15c_5}{8} \quad (71)$$

Integrating the right hand side of (67) accordingly, gives

$$-\frac{8}{3}x^4c_1 + \frac{32x^4c_2}{3} - 24x^4c_3 - 2x^3c_0 + 2x^3c_1 - 2x^3c_2 + 1144x^6c_4$$
$$-x^5c_0 + x^5c_1 - 9x^5c_2 + 49x^5c_3 - 161x^5c_4 - 384x^9c_4 + 1384x^9c_5 + \frac{2}{3}x^2c_1$$
$$-16x^8c_3 + 128x^8c_4 - 5968x^8c_5 - \frac{24x^7c_2}{5} + \frac{144x^7c_3}{5} - \frac{544x^7c_4}{3} + \frac{976x^7c_5}{3}$$
$$-\frac{391x^6c_3}{10} - 192x^{10}c_5 + \frac{116x^2c_2}{15} + 34x^3c_3 - \frac{242x^2c_3}{5} + \frac{512x^4c_4}{3}$$
$$-\frac{584x^4c_5}{3} + 1122x^3c_5 + \frac{5422x^2c_5}{21} - \frac{15}{2}x^5a_3 - x^4a_3 - x^2a_1 + \frac{512x^8a_5}{7}$$
$$+\frac{2}{3}x^4a_1 - 2x^4a_2 - \frac{3}{2}x^3a_2 + \frac{1}{2}x^3a_0 - \frac{1}{2}x^3a_1 + x^4 + x^2 - \frac{7363xc_5}{14} - \frac{319xc_4}{10}$$
$$+\frac{183xc_3}{10} - \frac{15}{2}xc_2 + \frac{3}{2}xc_1 - \frac{3}{2}xc_0 - 6 - 2x - 3x^2 - 4x^3 = 0 \quad (72)$$

To implement the Least Squares Method, equation (72) is squared and integrated from 0 to 1, to yield

$$1.077788878 - 0.2079365080c_0 - 0.09523809566c_2 - 50.58761903c_3$$
$$-925.2936401c_4 - 15405.39113c_5 + 0.02345679116c_1 + 37.77964406c_3$$
$$+683.1140196c_1 + 11515.20811c_1c_5 + 12.01386237c_2c_3 + 223.9020119c_2$$
$$+3711.344160c_2c_5 + 210179.6880c_3c_4 + 3565271.945c_3c_5 + 74121c_4c_5$$
$$-0.7071428571c_0 + 0.5371332371c_1 + 0.1665464163c_2 + 150.0266333c_3$$
$$+3034.717462c_4 + 51607.060c_5 + 0.1761904762{c_0}^2 + 0.1261022928{c_1}^2$$
$$+0.01768478435{c_2}^2 + 5239.917390{c_3}^2 + 2172124.269{c_4}^2$$
$$+633382934.3{c_5}^2 = 0 \quad (73)$$

Then, the partial derivatives of equation (73) with respect to $c_0, c_1, c_2 \ldots c_N$ are obtained as

$$\frac{\partial}{\partial c_0} = -0.079536500c_1 - 0.09523809566c_2 - 50.58761903c_3$$
$$-925.2936401c_4 - 15405.39113c_5 - 0.7071428571 + 0.3523809524c_0 = 0 \quad (74)$$

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$$\frac{\partial}{\partial c_1} = -0.2079365080c_0 + 0.02345679116c_2 + 37.77964406c_3$$
$$+683.1140196c_4 + 11515.20811c_5 + 0.5371332371 + 0.2522045856c_1 = 0 \quad (75)$$

$$\frac{\partial}{\partial c_2} = -0.09523809566c_0 + 0.02345679116c_1 + 12.01386237c_3$$
$$+223.9020119c_4 + 3711.344160c_5 + 0.1665464163 + 0.03536956870c_2 = 0 \quad (76)$$

$$\frac{\partial}{\partial c_3} = -50.58761903c_0 + 37.77964406c_1 + 12.01386237c_2$$
$$+210179.6880c_4 + 3565271.945c_5 + 150.0266333 + 10479.83478c_3 = 0 \quad (77)$$

$$\frac{\partial}{\partial c_4} = -925.2936401c_0 + 683.1140196c_1 + 223.9020119c_2$$
$$+210179.6880c_3 + 74121517.05c_5 + 3034.717462 + 4344248.538c_4 = 0 \quad (78)$$

$$\frac{\partial}{\partial c_5} = -15405.39113c_0 + 11515.20811c_1 + 3711.344160c_2$$
$$+3565271.945c_3 + 74121517.05c_4 + 51607.06019 + 1266765869.0c_5 = 0 \quad (79)$$

The equations (74) to (79) are solved simultaneously to obtain the constants values

$$c_0 = -0.1290929046e^{-2}, c_1 = -.1833865301, c_2 = -.2157377290,$$
$$c_3 = -0.8324142626e^{-2}, c_4 = 0.1544404392e^{-3}, c_5 = -0.2406448157e^{-4}$$

Substituting the values into the trial solution, gives the required approximate solution

$$y_5(x) = -0.3256300x^5 - 0.7986e^{-4}x^4 + 1.027063200x^3 + 1.06450000x^2$$
$$+0.679e^{-1}x + 0.1256500e^{-3}$$

Example 2: Least Square Method

Consider the fractional Volterra-Fredholm integro-differential equations.

$$D^\alpha y(x) = -x^2 - x^4 - x^6 + \int_0^x (3xt + 4x^2t^2)y(t)dt + \int_0^1 3xt^2 y(t)dt \quad x \in [0,1] \quad (80)$$

subject to initial conditions (ic)

$$y(0) = y'(0) = 0 \quad (81)$$

where $\alpha = 2.5$ and exact solution is $y(x) = x^2 + x^3$

Assuming

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$$y_5(x) = \sum_{j=0}^{5} c_j y(x) = c_0 + c_1(2x-1) + c_2(8x^2-8x+1)$$

$$+c_3(32x^3-48x^2+18x-1)+c_4(128x^4-256x^3+160x^2-32x+1) \quad a \le x \le b \qquad (82)$$

substituted (83) into (81), we have

$$D^{\alpha}\sum_{j=0}^{5} c_j y(x) = -x^2 - x^4 - x^6 + \int_0^x (3xt+4x^2t^2)\sum_{j=0}^{5} c_j y(t)dt + \int_0^1 3xt^2 \sum_{j=0}^{5} c_j y(t)dt, x \in [0,1] \qquad (83)$$

Applying $\alpha = \frac{5}{2}$ -derivative on the LHS of (84), gives

$$32768x^5c_5 + 4096x^4c_4 - 4096x^4c_5 + 384x^3c_3 - 3072x^3c_4 + 13440x^3c_5$$
$$+16x^2c_2 - 96x^2c_3 + 320x^2c_4 - 800x^2c_5 - xc_1 + 4xc_2 - 9xc_3 + 16xc_4$$
$$-25xc_5 + \frac{3}{4}c_0 - \frac{3}{4}c_1 + \frac{3}{4}c_2 - \frac{3}{4}c_3 + \frac{3}{4}c_4 - \frac{3}{4}c_5 \qquad (84)$$

Integrating the right hand side of (85) accordingly, gives

$$-\frac{160x^4a_4}{3} + \frac{317x^3a_4}{6} - 16x^2a_4 + xa_4 - 156x^5a_5 - \frac{64x^7a_4}{3} + \frac{640x^7a_5}{3}$$
$$-\frac{416x^6a_5}{3} - 4x^6a_3 + \frac{256x^6a_4}{5} - \frac{72x^5a_4}{5} - \frac{3}{2}x^5a_2$$
$$+\frac{15}{2}x^5a_3 + -\frac{1}{2}x^3a_0 + \frac{1}{2}x^3a_1 + x^2a_1 - \frac{512x^8a_5}{7} + a_0x + \frac{790x^4a_5}{3} - \frac{797x^3a_5}{6} + 25x^2a_5 \qquad (85)$$

further result on these yields;

$$35a_0x - \frac{1}{15}xa_4 + \frac{11}{30}a_4 - \frac{1}{2}a_0 - \frac{25}{6}a_1 + \frac{3}{42}a_5 \qquad (86)$$

Equation (86) and (87) are put into the problem (81) to give

$$6x^8c_3 + 128x^8c_4 - \frac{68x^8c_5}{7} - \frac{24x^7c_2}{5} + 49x^5c_3 - 161x^5c_4 - \frac{84x^9c_4}{2} + \frac{384x^9c_5}{3} + \frac{2}{3}x^2c_1$$

$$+\frac{4}{3}x^2c_0 6 + 49x^5c_3 - 161x^5c_4 - \frac{13x^9c_4}{7} + \frac{384x^5c_5}{3} + \frac{2}{3}x^2c_1 + \frac{4}{3}x^2c_0 - 16x^8c_3 + 128x^8c_4$$
$$-\frac{5968x^8c_5}{7} - \frac{24x^7c_2}{5} + x^{0.5}a_0 - x^{0.5}a_1 + x^{0.5}a_2 - x^{0.5}a_3 + x^{0.5}a_4 - x^{0.5}a_5 + \frac{64x^7a_4}{3}$$

$$-\frac{640x^7a_5}{3} + \frac{416x^6a_5}{3} + 4x^6a_3 - \frac{256x^6a_4}{5} + \frac{72x^5a_4}{5} + \frac{3}{2}x^5a_2 - \frac{15}{2}x^5a_3 - x^4a_3 + \frac{2}{3}x^4a_1$$

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$$-2x^4a_2 - \frac{3}{2}x^3a_2 + \frac{1}{2}x^3a_0 - 1/2x^3a_1 - x^2a_1 + \frac{512x^8a_5}{7} - 2a_0x - \frac{14xa_4}{15} + 156x^5a_5$$

$$-36.14146667a_4 + 13.54055000a_3 - 6.770275002a_2 + 1/2a_0 + 2.423425001a_1 + \frac{790x^4a_5}{3} + \frac{797x^3a_5}{6} - 25x^2a_5 + xa_5 + a_1x + 3x^2a_2 - xa_2 + \frac{19}{2}x^3a_3 - 6x^2a_3 + xa_3 + \frac{160x^4a_4}{3} - \frac{317x^3a_4}{6} + 16x^2a_4 + 14883a_5 + 3x^2 + 4x^3 + 2x = 0 \quad (87)$$

The source function, $f(x) = -x^2 - x^4 - x^6$ is included and after some simplifications accordingly gives

$$\frac{232x^2c_2}{15} - \frac{1192x^2c_3}{15} - c_5 + \frac{8504x^2c_4}{21} + x^5c_5 - \frac{16568x^2c_5}{21} - 96x^7c_4 + 240x^7c_5 + 6x^2c_1 - 6x^2c_0 + 16x^2x^2c_2 + \frac{1600x^3x^2c_5}{3} + 64x^3x^2c_3 - \frac{32x^3x^2c_2}{3} + 4x^2xc_5 - \frac{1024x^6x^2c_5}{3} - 4x^2xc_4 + 256x^4x^2c_4 + 4x^2xc_1 + 4x^2xc_3 - 4x^2xc_0 - 4x^2x^2c_1 - 4x^2xc_2 - \frac{512x^5x^2c_4}{5} - 36x^2x^2c_3 - 32x^4x^2c_3 - 100x^2x^2c_5 + 64x^2x^2c_4 + \frac{512x^5x^2c_5}{5} - \frac{384x^9c_4}{7} + \frac{384x^9c_5}{7} - \frac{2398x^2c_4}{5} + 102x^2c_3 - 22x^2c_2 + 8/3x^2c_0 + 128x^8c_4 - \frac{24x^7c_2}{5} + \frac{144x^7c_3}{5} - \frac{3}{2}x^6c_1 + 6x^6c_2 - x^5c_0 + x^5c_1 + 955x^2c_5 - 192x^{10}c_5 + 128x^4c_4 - 128t^4c_5 + 32x^3c_3 - 256x^3c_4 + 1120x^3c_5 + 8x^2c_2 - 48x^2c_3 + 160x^2c_4 + 2xc_1 - 8xc_2 + 18xc_3 - 32xc_4 + 50xc_5 + x^6 + x^4 + x^2 = 0 \quad (88)$$

To implement the Least Squares Method, equation (89) is squared and integrated from 0 to 1, to yield.

$$0.80853{c_2}^2 + 5.34738c_2c_5 + 8.14025c_3c_4 + 65.990953c_3c_5 + 20.693c_4c_5 + 93.933632c_2c_4 + 9.791860{c_3}^2 + 31.72002{c_4}^2 + 80.48{c_5}^2 + 1.0777888 - 3.4601175c_1c_4 - 52.811301c_1c_5 - 6.6055295c_1c_2 - 99.75176c_0c_5 + 1.557467532c_0c_1 - 23.19009965c_0c_2 - 4.6254307c_0c_3 + 0.25300918{c_1}^2 + 4.8860009c_4 + 62.3661185c_3 + 13.5669763c_2 - 3.266666667c_0 + 2.525829726{c_0}^2 = 0 \quad (89)$$

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Then, the partial derivatives of equation (90) with respect to $c_0, c_1, c_2 \ldots c_N$ are obtained as

$$\frac{\partial}{\partial c_0} = -19.75176c_5 - 8.0846015c_4 + 1.557467532c_1 - 23.1900996c_2$$
$$-24.625430c_3 - 3.2666667 + 5.051659452c_0 = 0 \quad (90)$$

$$\frac{\partial}{\partial c_1} = -75.407025c_3 - 23.4601175c_4 - 52.81131c_5 - 6.605529c_2$$
$$+1.5574675c_0 + 0.50601836c_1 - 1.0345090 = 0 \quad (91)$$

$$\frac{\partial}{\partial c_2} = 1.6170735c_2 + 5.34738c_5 + 2.511714c_3 + 93.933632c_4$$
$$-6.60552950c_1 - 23.19009965c_0 + 13.56697636 = 0 \quad (92)$$

$$\frac{\partial}{\partial c_3} = 8.14025c_4 + 5.9909c_5 + 2.51174c_2 + 19.58370c_3 - 75.40702c_1$$
$$-4.62543c_0 + 2.366112 = 0 \quad (93)$$

$$\frac{\partial}{\partial c_4} = 8.14025c_3 + 0.693c_5 + 93.933632c_2 + 3.4400c_4 - 23.46011c_1$$
$$-6.084601c_0 + 4.886000 = 0 \quad (94)$$

$$\frac{\partial}{\partial c_5} = 5.34738c_2 + 65.9909c_3 + 20.6932c_4 + 6.0961c_5 - 12.81130c_1$$
$$-19.7517c_0 + 32.52076 = 0 \quad (95)$$

The system of equations (91) to (96) are solved simultaneously to obtain the constants values

$$c_0 = 0.740319035, c_1 = 0.4171666414, c_2 = 0.1752582132,$$
$$c_3 = 0.11952398e^{-2}, c_4 = -0.41211550e^{-3}, c_5 = 0.42709384e^{-4}$$

Substituting the values into assume approximate solution, gives the required approximate solution

$$y_5(x) = -0.801975e^{-4}x^5 - 0.95600001e - 3x^4 + 0.90e - 3x^3 + 1.0000 * x^2$$
$$+6.0570x + 2.9875000 \quad (96)$$

Description of Akbari-Ganji Method

Consider the Volterra-Fredholm integro-differential equation

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$$D^{\alpha} y(x) + \lambda_1 \int_a^b k(x,t) y(t) dt + \lambda_2 \int_a^b k(u,t) y(t) dt = f(x) \quad x \in [a,b] \tag{97}$$

subject to conditions:

$$y^k(0) = \phi_k ; k = 1;2....n$$

and an assume an approximate solution

$$y_N(x) = \sum_{j=0}^{N} c_j y_j(x) = c_0 + c_1(2x-1) + c_2(8x^2 - 8x + 1)$$
$$+ c_3(32x^3 - 48x^2 + 18x - 1) + c_4(128x^4 - 256x^3 + 160x^2 - 32x + 1) \tag{98}$$

Substituting equation (99) into (98), gives:

$$D^{\sigma}\left(\sum_{i=0}^{N} a_i T_i^*(x)\right) + \lambda_1 \int_a^b k(x,t)\left(\sum_{i=0}^{N} a_i T_i^*(t)\right) dt$$
$$+ \lambda_2 \int_a^b k(x,t)\left(\sum_{i=0}^{N} a_i T_i^*(t)\right) dt = f(x) \qquad x \in [a,b] \tag{99}$$

We compute the derivatives of (100) with respect to α and β for α, $\alpha-1$, $\alpha-2$, $\ldots$ up to $\alpha-(n-k)$, we get:

$$D^{\alpha}\left(\sum_{i=0}^{N} a_i T_i^*(x)\right) + \lambda_1 \int_a^b k(x,t)\left(\sum_{i=0}^{N} a_i T_i^*(t)\right) dt + \lambda_2 \int_a^b k(x,t)\left(\sum_{i=0}^{N} a_i T_i^*(t)\right) dt = f(x) \qquad x \in [a,b] \tag{100}$$

$$D^{\alpha+1}\left(\sum_{i=0}^{N} a_i T_i^*(x)\right) + \lambda_1 \int_a^b k(x,t)\left(\sum_{i=0}^{N} a_i T_i^*(t)\right) dt + \lambda_2 \int_a^b k(x,t)\left(\sum_{i=0}^{N} a_i T_i^*(t)\right) dt = f(x) \qquad x \in [a,b] \tag{101}$$

$$D^{\alpha+(n-k)}\left(\sum_{i=0}^{N} a_i T_i^*(x)\right) + \lambda_1 \int_a^b k(x,t)\left(\sum_{i=0}^{N} a_i T_i^*(t)\right) dt + \lambda_2 \int_a^b k(x,t)\left(\sum_{i=0}^{N} a_i T_i^*(t)\right) dt = f(x) \qquad x \in [a,b] \tag{102}$$

evaluating equations (101) to (103) at the initial condition; $x = t = 0$ we get

$$D^{\alpha}\left(\sum_{i=0}^{N} a_i T_i^*(0)\right) + \lambda_1 \int_a^b k(x,t)\left(\sum_{i=0}^{N} a_i T_i^*(0)\right) dt + \lambda_2 \int_a^b k(x,t)\left(\sum_{i=0}^{N} a_i T_i^*(0)\right) dt = f(0) \qquad x \in [a,b] \tag{103}$$

$$D^{\alpha+1}\left(\sum_{i=0}^{N} a_i T_i^*(0)\right) + \lambda_1 \int_a^b k(x,t)\left(\sum_{i=0}^{N} a_i T_i^*(0)\right) dt + \lambda_2 \int_a^b k(x,t)\left(\sum_{i=0}^{N} a_i T_i^*(0)\right) dt = f(0) \qquad x \in [a,b] \tag{104}$$

$$D^{\alpha+(n-k)}\left(\sum_{i=0}^{N} a_i T_i^*(0)\right) + \lambda_1 \int_a^b k(x,t)\left(\sum_{i=0}^{N} a_i T_i^*(0)\right) dt + \lambda_2 \int_a^b k(x,t)\left(\sum_{i=0}^{N} a_i T_i^*(0)\right) dt = f(0) \qquad x \in [a,b] \tag{105}$$

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The remaining k-equations are obtained substituting the initial conditions into the trial solution. Thus, a system of $n+k$ equations is achieved which are then solved to find the values of the unknown constants involved.

Example 1: Akbari Ganji Method

Consider the fractional Volterra-Fredholm integro-differential equations.

$$D^{\alpha}u(x) = -6-2x-3x^2-4x^3+\int_0^x (1-xt)y(t)dt+\int_0^1 (x-t)y(t)dt \tag{106}$$

subject to initial conditions (ic)

$$y(0)=2, y'(0)=6y''(0)=24 \tag{107}$$

where $\alpha = \frac{7}{2}$ and exact solution is $y(x)=2+6x+12x^2$

We assume

$$y_5(x)=\sum_{j=0}^{5} c_j y(x) = c_0+c_1(2x-1)+c_2(8x^2-8x+1)+c_3(32x^3-48x^2$$
$$+18x-1)+c_4(128x^4-256x^3+160x^2-32x+1)+a\le x\le b \tag{108}$$

and substituting (109) into (107), we have

$$D^{\alpha}\sum_{j=0}^{5} c_j y(x) = -6-2x-3x^2-4x^3++\int_0^x (1-2xt)\sum_{j=0}^{5} c_j y(t)dt+\int_0^1 (x-t)\sum_{j=0}^{5} c_j y(t)dt, x\in[0,1] \tag{109}$$

Applying $\alpha = \frac{7}{2}$ -derivative on the LHS of (107), gives

$$81920x^5c_5+6144x^4c_4-6144x^4c_5+192x^3c_3-1536x^3c_4+6720x^3c_5-8x^2c_2$$
$$+48x^2c_3-160x^2c_4+400x^2c_5+3/2c_1x-6xc_2+\frac{27xc_3}{2}-24xc_4+\frac{75xc_5}{2}-\frac{15c_0}{8}$$
$$+\frac{15c_1}{8}-\frac{15c_2}{8}+\frac{15c_3}{8}-\frac{15c_4}{8}+\frac{15c_5}{8} \tag{110}$$

Integrating the right-hand side of (110) accordingly, gives

$$-\frac{64x^7a_4}{3}+\frac{640x^7a_5}{3}-\frac{416x^6a_5}{3}-4x^6a_3+\frac{256x^6a_4}{5}-\frac{72x^5a_4}{5}-\frac{3}{2}x^5a_2+\frac{15}{2}x^5a_3$$
$$+x^4a_3-\frac{3}{2}x^4a_1+2x^4a_2+\frac{3}{2}x^3a_2-\frac{}{2}x^3a_0+\frac{1}{2}x^3a_1+x^2a_1-\frac{512x^8a_5}{7}+a_0x$$
$$+\frac{790x^4a_5}{3}-\frac{797x^3a_5}{6}+25x^2a_5-xa_5-a_1x-3x^2a_2+xa_2-\frac{19}{2}x^3a_3+6x^2a_3-xa_3$$
$$-\frac{160x^4a_4}{3}+\frac{317x^3a_4}{6}-16x^2a_4+xa_4-156x^5a_5+49x^5c_3-161x^5c_4-384x^9c_4$$
$$+1384x^9c_5+\frac{2}{3}x^2c_1+\frac{3}{4}x^2c_0-16x^8c_3+128x^8c_4-5968x^8c_5-\frac{24x^7c_2}{5}+\frac{144x^7c_3}{5}$$

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$$-\frac{544x^7c_4}{3}+\frac{976x^7c_5}{3}-\frac{3}{2}x^6c_1+6x^6c_2-391x^6c_3-192x^{10}c_5+116x^2c_2+34x^3c_3-242x^2c_3$$
$$+512x^4c_4-258x^3c_4+3356x^2c_4+91x^5c_5-584x^4c_5+1122x^3c_5+5422x^2c_5-c_5$$
$$+c_4-c_3+c_2+c_0-c_1+x^6+\frac{3}{2}x^5a_2-\frac{15}{2}x^5a_3-x^4a_3-x^2a_1+512x^8a_5-2a_0x$$
$$+\frac{2}{3}x^4a_1-2x^4a_2-\frac{3}{2}x^3a_2+\frac{1}{2}x^3a_0-\frac{1}{2}x^3a_1-6-2x-3x^2-4x^3=0 \quad (111)$$

Now to implement the Akbari-Ganji Method, equation (112) is differentiated successively at α, $\alpha+1$, $\alpha+2$, L $\alpha+(n-2)$, and evaluated at $x=0$ to yield

$$3.70980431c_5+3.48986504c_4-3.708174c_3+3.702489814c_2-3.702489814c_1$$
$$+3.702489814c_0=0 \quad (112)$$

$$0.004730948-15.95510361c_5+22.12852613c_4-9.53268952c_3+5.666056937c_2$$
$$-4.125549458c_1+3.70177400c_0=0 \quad (113)$$

$$9.08223902+48.1946379c_5+101.0115149c_4-21.28880173c_3+8.1791735c_2$$
$$-4.543300c_1+3.683060780c_0=0 \quad (114)$$

$$0.77074555+8.8914792c_5+390.9745493c_4-41.49016626c_3+11.27488129c_2$$
$$-4.922122580c_1+3.541869449c_0=0 \quad (115)$$

$$2.120046747+208.95978c_5+109.204957c_4-72.75732686c_3+15.14316741c_2$$
$$-5.1907529c_1+2.8903584c_0=0 \quad (116)$$

$$1.16038418+64804.70365c_5+2339.644636c_4-117.7276052c_3+19.93559075c_2$$
$$-5.484864945c_1+0.702489814c_0=0 \quad (117)$$

The equations (113) to (118) are solved simultaneously to obtain the constants values

$$c_0=-.198351237, c_1=-.321622992, c_2=-.160619601, c_3=-0.409202428e^{-1},$$
$$c_4=-0.357303889e^{-2}, c_5=, c_5=-5.9154045210^{(-7)}$$

Substituting the above values into the trial solution, gives the required approximate solution

$$y_5(x)=-0.3293e^{-6}x^5-0.6540934e-4x^4-0.231549e-5x^3+1.087500034169732x^2$$
$$+1.05883595x-0.658753e-8 \quad (118)$$

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Example 2: Akbari-Ganji Method Consider the fractional Volterra-Fredholm integro-differential equations.

$$D^{\alpha}y(x) = -x^2 - x^4 - x^6 + \int_0^x (3xt + 4x^2t^2)y(t)dt + \int_0^1 3xt^2 y(t)dt \qquad x \in [0,1] \tag{119}$$

subject to initial conditions (ic)

$$y(0) = y'(0) = 0 \tag{120}$$

where $\alpha = 2.5$ and exact solution is $y(x) = x^2 + x^3$

Assuming

$$y_5(x) = \sum_{j=0}^{5} c_j y(x) = c_0 + c_1(2x-1) + c_2(8x^2 - 8x + 1) + c_3(32x^3 - 48x^2 + 18x - 1)$$

$$+c_4(128x^4 - 256x^3 + 160x^2 - 32x + 1) \; a \le x \le b \tag{121}$$

substituted (122) into (120), we have

$$D^{\alpha}\sum_{j=0}^{5} c_j y(x) = -x^2 - x^4 - x^6 + \int_0^x (3xt + 4x^2t^2)\sum_{j=0}^{5} c_j y(t)dt + \int_0^1 3xt^2 \sum_{j=0}^{5} c_j y(t)dt \quad x \in [0,1] \tag{122}$$

Applying $\alpha = \frac{5}{2}$ -derivative on the LHS of (123), gives

$$32768x^5c_5 + 4096x^4c_4 - 4096x^4c_5 + 384x^3c_3 - 3072x^3c_4 + 13440x^3c_5 + 16x^2c_2$$

$$-96x^2c_3 + 320x^2c_4 - 800x^2c_5 - xc_1 + 4xc_2 - 9xc_3 + 16xc_4 - 25xc_5 + \frac{3}{4}c_0 - \frac{3}{4}c_1 + \frac{3}{4}c_2 - \frac{3}{4}c_3 + \frac{3}{4}c_4 - \frac{3}{4}c_5 \tag{123}$$

Integrating the right hand side of (124) accordingly, gives

$$-xa_5 - a_1x - 3x^2a_2 + xa_2 - 19/2x^3a_3 + 6x^2a_3 - xa_3 - \frac{160x^4a_4}{3} + \frac{317x^3a_4}{6} - 16x^2a_4$$

$$+xa_4 - 156x^5a_5 - \frac{64x^7a_4}{3} + \frac{640x^7a_5}{3} - \frac{416x^6a_5}{3} - 4x^6a_3 + \frac{256x^6a_4}{5}$$

$$-\frac{72x^5a_4}{5} - 3/2x^5a_2 + 15/2x^5a_3 + x^4a_3 - \frac{2}{3}x^4a_1 + 2x^4a_2 + \frac{3}{2}x^3a_2 - \frac{1}{2}x^3a_0 + \frac{1}{2}x^3a_1$$

$$+x^2a_1 - \frac{512x^8a_5}{7} + a_0x + \frac{790x^4a_5}{3} - \frac{797x^3a_5}{6} + 25x^2a_5 \tag{124}$$

and

$$35a_0x - 1/15xa_4 + 11/30a_4 - 1/2a_0 - 25/6a_1 + 3/42a_5 \tag{125}$$

Equation (125) and (126) are put into the problem (120) to give

$$6x^8c_3 + 128x^8c_4 - \frac{68x^8c_5}{7} - \frac{24x^7c_2}{5} + 49x^5c_3 - 161x^5c_4 - \frac{84x^9c_4}{2} + \frac{384x^9c_5}{3}$$

$$+\frac{2}{3}x^2c_1 + \frac{4}{3}x^2c_0 6 + 49x^5c_3 - 161x^5c_4 - \frac{13x^9c_4}{7} + \frac{384x^5c_5}{3} + \frac{2}{3}x^2c_1 + \frac{4}{3}x^2c_0$$

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$$-16x^8c_3+128x^8c_4-\frac{5968x^8c_5}{7}-\frac{24x^7c_2}{5}+\frac{64x^7a_4}{3}-\frac{640x^7a_5}{3}+\frac{416x^6a_5}{3}$$
$$+4x^6a_3-\frac{256x^6a_4}{5}+\frac{72x^5a_4}{5}+\frac{3}{2}x^5a_2-\frac{15}{2}x^5a_3-x^4a_3+\frac{1}{2}x^4a_1-2x^4a_2-\frac{3}{2}x^3a_2$$
$$+\frac{1}{2}x^3a_0-\frac{1}{2}x^3a_1-x^2a_1+\frac{512x^8a_5}{7}-2a_0x--\frac{14xa_4}{15}+156x^5a_5-36.14146667a_4$$
$$+13.54055000a_3-6.770275002a_2+1/2a_0+2.423425001a_1+56.395$$
$$\frac{790x^4a_5}{3}+\frac{797x^3a_5}{6}-25x^2a_5+xa_5+a_1x+3x^2a_2-xa_2+19/2x^3a_3-6x^2a_3+xa_3$$
$$+\frac{160x^4a_4}{3}-\frac{317x^3a_4}{6}+16x^2a_4 14883a_5+3x^2+4x^3+2x=0 \tag{126}$$

The source function, $f(x)=-x^2-x^4-x^6$ is included and after some simplifications accordingly gives

$$\frac{232x^2c_2}{15}-\frac{1192x^2c_3}{15}-c_5+\frac{8504x^2c_4}{21}+x^5c_5-\frac{16568x^2c_5}{21}-96x^7c_4+240x^7c_5$$
$$-\frac{75x^6c_5}{2}-\frac{27x^6c_3}{2}+24x^6c_4-x^5c_4-x^5c_2+x^5c_3-\frac{8}{3}x^2c_1-560x^8c_5+\frac{32x^3x^2c_2}{3}+4x^2xc_5$$
$$-1120x^4x^2c_5-\frac{1024x^6x^2c_5}{3}-4x^2xc_4+256x^4x^2c_4+4x^2xc_1+4x^2xc_3-4x^2xc_0$$
$$-4x^2x^2c_1-4x^2xc_2-\frac{512x^5x^2c_4}{5}-36x^2x^2c_3-32x^4x^2c_3-100x^2x^2c_b5$$
$$+64x^2x^2c_4-\frac{640x^3x^2c_4}{3}+\frac{512x^5x^2c_5}{5}-\frac{384x^9c_4}{7}+\frac{384x^9c_5}{7}-\frac{2398x^2c_4}{5}+102x^2c_3$$
$$-22x^2c_2+\frac{8}{3}x^2c_0-16x^8c_3+128x^8c_4-\frac{24x^7c_2}{5}+\frac{144x^7c_3}{5}-\frac{3}{2}x^6c_1+6x^6c_2+955x^2c_5$$
$$-192x^{10}c_5+512x^5c_5+128x^4c_4-128t^4c_5+32x^3c_3-256x^3c_4+1120x^3c_5+8x^2c_2$$
$$-48x^2c_3+160x^2c_4-400x^2c_5+2xc_1-8xc_2+18xc_3-32xc_4+50xc_5+x^6+x^4+x^2=0 \tag{127}$$

To implement the Akbari-Ganji Method, equation (128) is differentiated successively at α, $\alpha+1$, $\alpha+2$, L $\alpha+(n-2)$, and evaluated at $x=0$ to yield

$$-6.15567381c_5-0.450681555c_4+0.105291280c_3-0.157219440c_2+0.133370433c_0$$

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$$-0.13677600c_1 + 0.041664 = 0 \tag{128}$$

$$-7.3877323c_5 + 0.52472735c_4 + 0.906513560c_3 - 0.048619320c_2 + 0.0945233333c_0$$
$$-0.5038440000c_1 + 0.14897696 = 0 \tag{129}$$

$$1.3413506c_5 + 0.42636544c_4 - 0.86693248c_3 - 0.987313280c_2 + 0.072454000c_0$$
$$+0.2356760000c_1 + 0.53456256 = 0 \tag{130}$$

$$5.6245501c_5 - 2.2709180c_4 - 0.359647667c_3 + 0.951104000c_2 - 0.983467667c_0$$
$$+0.928624399c_1 + 1.3181744 = 0 \tag{131}$$

$$1.1375302c_5 + 0.5529643c_4 + 0.58854246c_3 - 0.65165040c_2 - 0.68734635c_0$$
$$+0.40639284397c_1 + 3.371917424 = 0 \tag{132}$$

$$6.375302c_5 + 3.54529643c_4 + 4.6587854246c_3 - 5.695165040c_2 - 0.6458734635c_0$$
$$+0.04634284397c_1 + 2.361917424 = 0 \tag{133}$$

The equations (130) to (134) are solved simultaneously to obtain the constants values

$$c_0 = 32.6334712\ c_1 = 5.0572425 \quad c_2 = 8.8191650 \quad c_3 = 3.92923565$$
$$c_4 = 6.54938014 c_5 = -4.2782329878$$

Substituting the values into the trial solution, gives the required approximate solution

$$y_5(x) = -0.89986e - 5x^4 + 1.057663200x^3 + 1.07450000x^2 + 0.879e - 1x0.26450000000e^{-1} \tag{134}$$

TABLES AND GRAPHS

The Tables and Graphs of the results obtained are presented below

Table 1: Error of Results for Example 1

x	Exact	SCM	Error	LSM	Error	AGM	Error
0.0	0.0000000000	-0.000000013	1.304e-08	-0.000000013	1.304e-08	-0.00000013	1.304e-08
0.1	0.011000000	0.012115300	1.60e-08	0.01100000	2.500e-08	0.01000215	1.909e-07
0.2	0.0480000000	0.0480000022	2.274e-07	0.0480000024	2.41e-08	0.0480000013	1.317e-06
0.3	0.1170000000	0.011699876	1.200e-07	0.011699876	1.260e-07	0.01169760	1.300e-06

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x	Exact	SCM	Error	LSM	Error	AGM	Error
0.4	0.2240000000	0.2646835890	4.0684e-06	0.2646835890	4.0684e-07	0.26000415	4.563e-06
0.5	0.3750000000	0.4213556260	4.6356e-06	0.4213556260	4.6356e-06	0.25000641	2.491e-06
0.6	0.5760000000	0.6250281228	4.9028e-06	0.6250281228	4.9028e-06	0.3600093	3.317e-06
0.7	0.8330000000	0.8811010819	4.8101e-05	0.8811010819	4.8101e-05	0.49001273	1.740e-05
0.8	1.1520000000	1.1949745060	4.2975e-05	1.1949745060	4.2975e-05	1.64001663	1.653e-05
0.9	1.5390000000	1.5720483960	3.3048e-04	1.5720483960	3.3048e-04	1.81002104	2.184e-05
1.0	2.0000000000	2.0177227570	1.7723e-04	2.0177227570	1.7723e-04	1.90002602	2.021e-05

Table 2: Error of Results for Example 2

x	Exact	SCM	Error	LSM	Error	AGM	Error
0.0	2.0000000000	2.0000000050	5.0000e-09	2.0000000050	5.0000e-09	0.000003	3.0646e-08
0.1	2.6100000000	2.6098219220	1.7808e-08	2.6098219220	1.7808e-08	1.010049	1.5406e-07
0.2	3.2400000000	3.2396446890	3.5531e-08	3.2396446890	3.5531e-08	1.040798	2.7390e-06
0.3	3.8900000000	3.8894682630	5.3174e-07	3.8894682630	5.3174e-07	1.094201	1.2886e-06
0.4	4.5600000000	4.5592925710	7.0743e-07	4.5592925710	7.0743e-07	1.173184	3.5849e-05
0.5	5.2500000000	5.2491175100	8.8249e-07	5.2491175100	8.8249e-07	1.284621	1.1521e-05
0.6	5.9600000000	5.9589429540	1.0570e-06	5.9589429540	1.0570e-06	1.432638	1.3014e-05
0.7	6.6900000000	6.6887687440	1.2313e-06	6.6887687440	1.2313e-06	1.630178	1.4008e-04
0.8	7.4400000000	7.4385946960	1.4053e-05	7.4385946960	1.4053e-05	1.897739	1.64472e-04
0.9	8.2100000000	8.2084205950	1.5794e-05	8.2084205950	1.5794e-05	2.235927	2.44016e-03
1.0	9.0000000000	8.9982462010	1.7538e-05	8.9982462010	1.7538e-05	2.716913	1.38147e-02

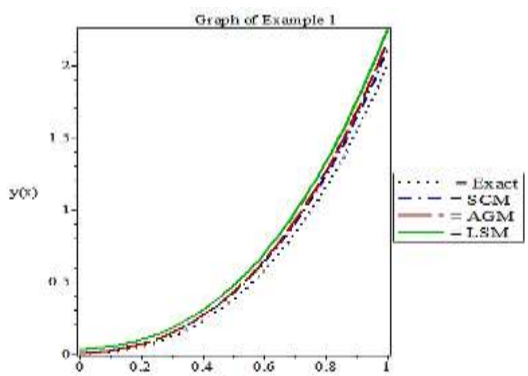

Fig. 1. Comparison of Results for exact, sch, lsm and agm for example 1

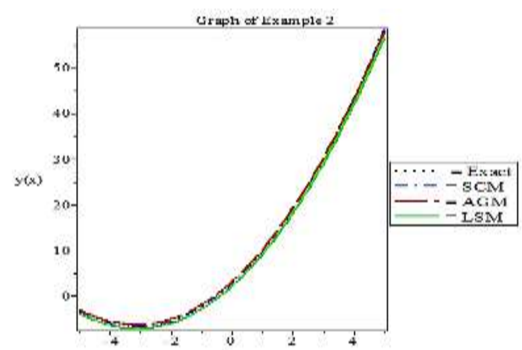

Fig. 2. Comparison of Results for exact, sch, lsm and agm for example 2

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DISCUSSION OF RESULTS

This research presents, some classes of Fractional Volterra-Fredholm Integro-differential Equations were solved by Standard Collocation, Least Squares and Akbari-Ganji Methods. Three examples were considered and the proposed methods results are impressive.

Example 1 was solved at $\alpha = \frac{9}{2}$ taking $n = 5$. The first two examples from Standard Collocation Method an approximation result of ;

$$-0.916845e^{-6}x^4 - 0.9995e^{-4}x^4 + 0.9890596x^3 + 1.0048500x^2 + 0.6850x + 0.12100e^{-3}$$

was obtained and in comparison, with the exact solution who's the results converged. Same for Example 2 was solved at $\alpha = \frac{5}{2}$ taking $n = 5$ For Standard Collocation Method, an approximate

Least Squares and Akbari-Ganji methods for their first examples as shown on the graphs.
Result of ;

$$0.00000087650x^5 - 0.64300001e - 4x^4 + 0.501435e - 2x^3 + 1.0000x^2 + 6.045x + 2.598657 = 0$$

was obtained and in comparison, to the exact solution, the results converged. Same for Least Squares and Akbari-Ganji methods as shown on the graphs.

CONCLUSION

Some classes of Fractional Volterra-Fredholm Integro-differential equations were solved by Standard Collocation, Least Squares and Akbari-Ganji Methods. Two examples were solved for each method to demonstrate the three proposed methods. From the tables and the graphs, it is clear that the methods performed very well. The results converge rapidly to the exact solutions. It is therefore concluded that the methods are effective and accurate in solving the class of problem considered. It is however noticed that Least Squares method performed slightly better than the Standard Collocation Method.

The modest contributions of this research are in line with that some orthogonal polynomials valid in the interval [0,1] have been successively adopted and used for the solutions of fractional Volterra-Fredholm integro-differential equations, fractional Volterra-Fredholm integro-differential equations have been obtained using Standard Collocation Method procedures, fractional Volterra-Fredholm integro-differential have been achieved using Least Squares Method algorithm and fractional Volterra-Fredholm integro-differential have been gotten using Akbari-Ganji Method algorithm. For further study, this research work can be extended to multi-order fractional partial differential equations.

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